Essence of Linear Algebra Notes

From 3b1b lectures

Evan Pratten

Contents

Eigen vectors & values

- The eigen vector(s) of a transformation matrix is the vector(s) that are not moved by the transformation.
- During a 3D rotation, the eigen vector falls along the axis of rotation
- A transform of $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ would result in $\hat{j}$ being the eigen vector

Solving

Given the formula

$$A\vec{v} = \lambda\vec{v}$$

where A is a matrix, and λ is a scalar, we must multiply λ by an identity matrix I to properly solve.

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \lambda I$$

$$A\vec{v} = (\lambda I)\vec{v}$$

$$A\vec{v} - (\lambda I)\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

Checking if λ is an eigen value

Let's say we want to check if λ is an eigen value of

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

All we need to do is check if the following is true

$$det(A - \lambda) == 0$$

This can be solved by:

$$dat(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}) = 0 = (3-\lambda)(2-\lambda) - (1)(0)$$

Vector space & Null space

Rank

Rank	Meaning
Full Rank	Rank number = column count of matrix
Rank 1	Transforms to a line
Rank 2	Transforms to a plane
Rank 3	Transforms to a cube
...	...

- Column space means:
- All possible values in $A\vec{v}$
- Null space (kernel) means:
- All values a long a line or plane that get "squished" to $(0, 0)$ after a transform to a lower rank

Rewriting a system of equations as a matrix

Say we have the following system

$$2x + 5y + 3z = -3$$
$$4x + 8x = 0$$
$$1x + 3y = 2$$

If we subs in some 0s, we can line up all the variables

$$2x + 5y + 3z = -3$$
$$4x + 0y + 8z = 0$$
$$1x + 3y + 0z = 2$$

Then convert it to a matrix

$$\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

This matrix is layed out in the form of $A\vec{x} = \vec{v}$. To solve for all variables, we just need to do the following:

$$A^{-1}A\vec{x} = A^{-1}\vec{v}$$
$$\vec{x} = A^{-1}\vec{v}$$

The determinant

A measurement how much a linear transformation stretches an area or volume.

If we take a unit vector, and transform it by only $\hat{\imath}$ and $\hat{\jmath}$:

$$det(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}) = 6$$

The resulting vector's area has grown by 6. This is the determinant. Now, look at a simple transform that only shears:

$$det(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) = 1$$

Since there is no scaling of $\hat{\imath}$ or $\hat{\jmath}$, the determinant remains 1.

Calculating the determinant of non-squares

Like a derivative, we can simply fill an irregular / curvy shape with small squares to the desired precision to determine its determinant.

Determinant of 0

If a transform goes down a dimension, the determinant will always be 0 (since the lower dimension removes one of the required axes to measure volume in the original dimension)

Other notes

- Any transform that looks like it "flips over" is called "inverting orientation". If this happens, the determinant will always be less than 0. But $abs(det(...))$ is still the correct scaling factor
- As the angle between $\hat{\imath}$ and $\hat{\jmath}$ approaches 0°, the determinant gets closer to 0. Once it passes, it get negative.
- If the "right hand rule" stops working after a transformation, the determinant has become negative

Calculating the determinant of a 2D linear transform

$$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$$

Interesting property

Here is an interesting property of the determinant of two matrices (m_1 and m_2) that grant raised:

$$det(m_1 \cdot m_2) = det(m_1) \cdot det(m_2)$$

Cross products

If we have two vectors, $\vec{v}$ and $\vec{w}$, and we use them as two connecting sides of a parallelogram, the area of the resulting parallelogram is actually the cross product between $\vec{v}$ and $\vec{w}$ (see figure 1). This operation follows the right-hand-rule, and the result is actually a vector along the thumb (where $\vec{v}$ is the pointer finger and $\vec{w}$ is the middle finger)

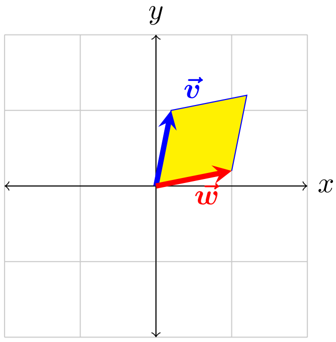

Figure 1: The cross product as area

Computing the scalar length of a cross product

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \end{bmatrix} = det(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix})$$

An example calculation

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = det(\begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}) = -3 \cdot 1 - 2 \cdot 1 = -5$$

$$(3\vec{v}) \times \vec{w} = 3(\vec{v} \times \vec{w})$$

Computing the actual vector of a cross product

The cross product of two vectors is not actually a number, but a vector in the 3rd dimension. Here is how to calculate this vector:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = det(\begin{bmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix})$$

Given this, the following calculation is true. See figure 2 for a 3D representation of this equation:

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

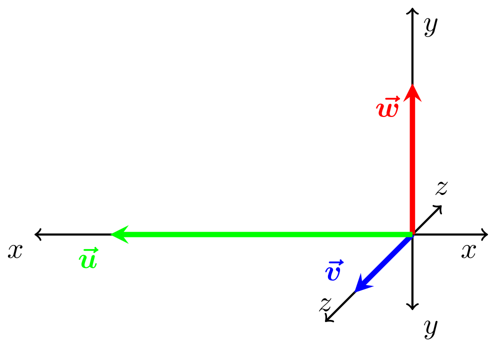

Figure 2: The cross product as vector $\vec{u}$

Dot products & duality

$$\vec{v} \cdot \vec{w} = \vec{u}$$

The dot product ($\vec{u}$) of the two vectors $\vec{v}$ and $\vec{w}$ is a scalar that follows the right-hand-rule, where $\vec{v}$ would be your pointer finger, and $\vec{w}$ would be your middle finger.

> duality roughly refers to "natural but surprising correspondence"

Calculating the dot of two vectors

The dot product of these vectors is the dots of the individual components, added together

$$\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \\ 8 \end{bmatrix} = 2 \cdot 8 + 7 \cdot 2 + 1 \cdot 8$$

The sign of the dot

The sign of the dot product of $\vec{v}$ and $\vec{w}$ is determined by the relative positioning of the two vectors to eachother. This can simply be determined via the right hand rule. Although, if $\vec{v}$ and $\vec{w}$ are perfectly perpendicular to eachother, the dot product will be 0.

Interesting properties

$$2\vec{v} \cdot \vec{w} = 2(\vec{v} \cdot \vec{w})$$

A visual explication

To visually determine the dot product of $\vec{v}$ and $\vec{w}$, cast $\vec{v}$ on to $\vec{w}$, then multiply the length of the casted vector ($\vec{u}$) by the length of $\vec{w}$ to get the result. This is demonstrated in figure 3.

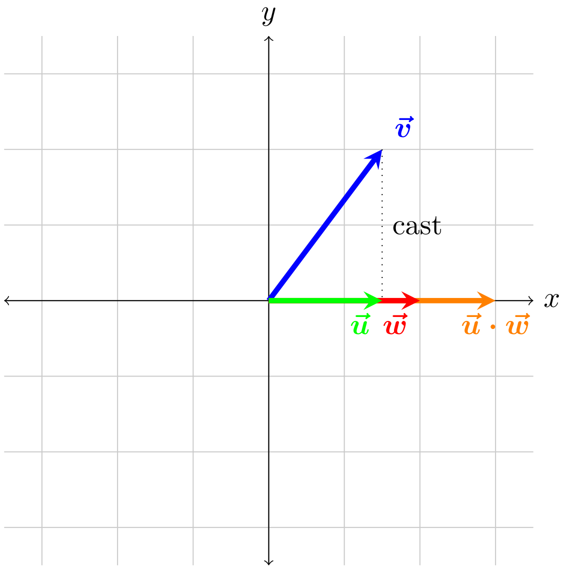

Figure 3: The dot product as a casted vector

Duality

If we have a linear transformation from the 2nd dimension down to the 1st, $\vec{w}$, this matrix can be flipped on it's side (effectively making a vector), and this new vector $\vec{u}$ is the "dual vector", and can be dotted with a vector $\vec{v}$ to get the same result as the transformation.

$$\vec{w} = \begin{bmatrix} 3 & 1 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{w}\vec{v} = \vec{u} \cdot \vec{v}$$

Non-Square Matrices

This is a small note on non-square matrices.

- Each column of the matrix is a "hat"
- They are used to transform between dimensions
- An x-by-y matrix transforms from the x-dimension to the y-dimension

An example

Take a look at the following matrix

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}$$

This is a 3-by-2 matrix, so it is used to transform from the 3rd dimension to the 2nd dimension